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Periodicity and Chaos in a Modulated Logistic Map

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We study the onset of chaos in a logistic map whose parameter is modulated nonlinearly. The bifurcation pattern with respect to a parameter μ is obtained and the critical value of μ is seen to be 0.89, where periodicity just ends. Further evidence for this regime is obtained from the analysis of the intermittency pattern. The stability in the different ranges of μ under repeated iteration is exhibited by the values of Lyapunov exponents. Beyond $\mu = 0.89$, the largest Lyapunov exponent becomes positive and the situation turns out to be unstable. Confirmation comes from a functional analysis of the stable and unstable manifolds which touch at $\mu = 0.89$.

1. INTRODUCTION

The study of nonlinear maps in order to analyze the behavior of chaotic phenomena started after the pioneering work of Feigenbaum (1980). Various kinds of nonlinear maps have been studied and more interesting features have been extracted. Among various nonlinear systems are the logistic map (May, 1976), the standard map (Chirikov, 1979), and the quadratic map (Collet and Eckmann, 1980), which have paved the way for the understanding of such intricate behavior. Interest has grown in higher-dimensional systems. A simple method of generating a higher-dimensional system is to couple low-dimensional nonlinear mappings. It is usually conjectured that when such coupling occurs (e.g., in a coupled logistic lattice, a coupled neuron-like lattice the dynamical behavior becomes much more complex than in a single-map system and the symbolic dynamics exhibits a behavior similar to cellular automata (Collet and Eckmann, 1980). In this paper we study a higher-dimensional system obtained by modulating the original parameter of a logistic map nonlinearly. Our analysis explores the road to bifurcation that leading to intermittency, the approach to instability via

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Lyapunov exponents, and the homoclinic tangency of stable–unstable manifolds, away from the fixed point.

2. FORMULATION

The mapping we consider is a logistic one, whose parameter is also modulated nonlinearly via another map and is written as

$$
\begin{aligned}
X_{n+1} &= 4\lambda_n x_n(1-x_n) \\
\lambda_{n+1} &= 4\mu\lambda_n(1-\lambda_n)
\end{aligned}
\tag{1}
$$

The 1-cycle fixed point of this 2D map is given by

$$
x^* = \lambda^* = \frac{3\mu - 1}{4\mu - 1} = a \quad \text{(say)} \tag{2}
$$

If we linearize the mapping with respect to this fixed point, that is, we set

$$
\begin{aligned}
x_n &= x^* + \delta x_n \\
\lambda_n &= \lambda^* + \delta\lambda_n
\end{aligned}
\tag{3}
$$

then we obtain

$$
\begin{pmatrix} \delta x_{n+1} \\ \delta\lambda_{n+1} \end{pmatrix} = \begin{pmatrix} 4(1-x^*)\lambda^* - 4\lambda^* x^* & 4(1-x^*)x^* \\ 0 & 4\mu(1-2\lambda^*) \end{pmatrix} \begin{pmatrix} \delta x_n \\ \delta\lambda_n \end{pmatrix} \tag{4}
$$

So the linear stability of the system can be analyzed by analyzing the eigenvalues of the matrix, which is nothing but the Jacobian matrix of the original equation (1). Since in the sequel we will be using this in our discussion of the Lyapunov exponents in a more elaborate analysis, we do not explore it here.

Instead we go over to the discussion of the bifurcation diagram drawn on a computer which results from repeated iteration of (1). Figures 1a and 1b plot the resultant values of x_n versus values of the new parameter μ. The two figures show two different ranges of values of μ. In Figure 1a the minimum value of μ is 0.7 and it exhibits the bifurcation pattern for $0.7 \leq \mu \leq 0.9$. In both figures the situation is similar to the usual logistic pattern, but with some asymmetry. The iterated values follow two different paths starting from $\mu = 0.75$. Afterward there are repeated bifurcations and finally we arrive at a chaotic scene at $\mu \gtrsim 0.89$, where the ordered structure ceases to exist. To confirm our observation, we next plot the intermittency curve for the modulated map. In this case some value is preassigned and repeated iterated values of (x_n, λ_n) are obtained. We have plotted these values of x_n against n. For example, in Figure 2b, where $\mu = 0.89$, after some initial disorderliness, the map shows periodic character. But as μ is

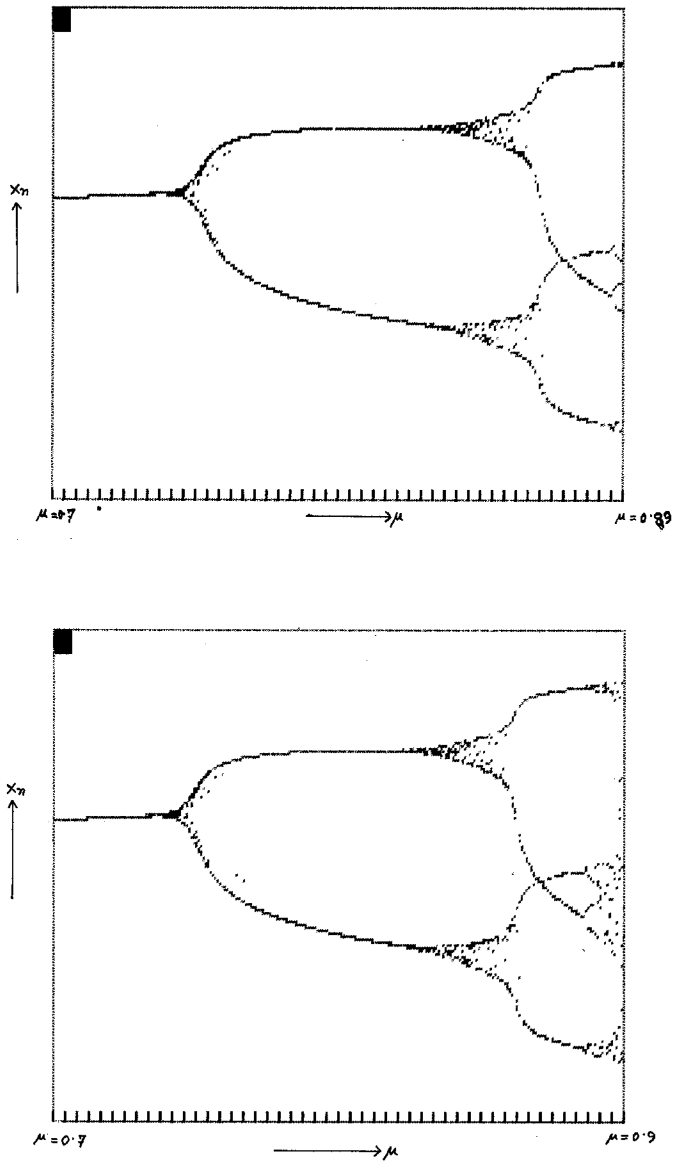


Fig. 1. Bifurcation diagram for the modulated logistic map.

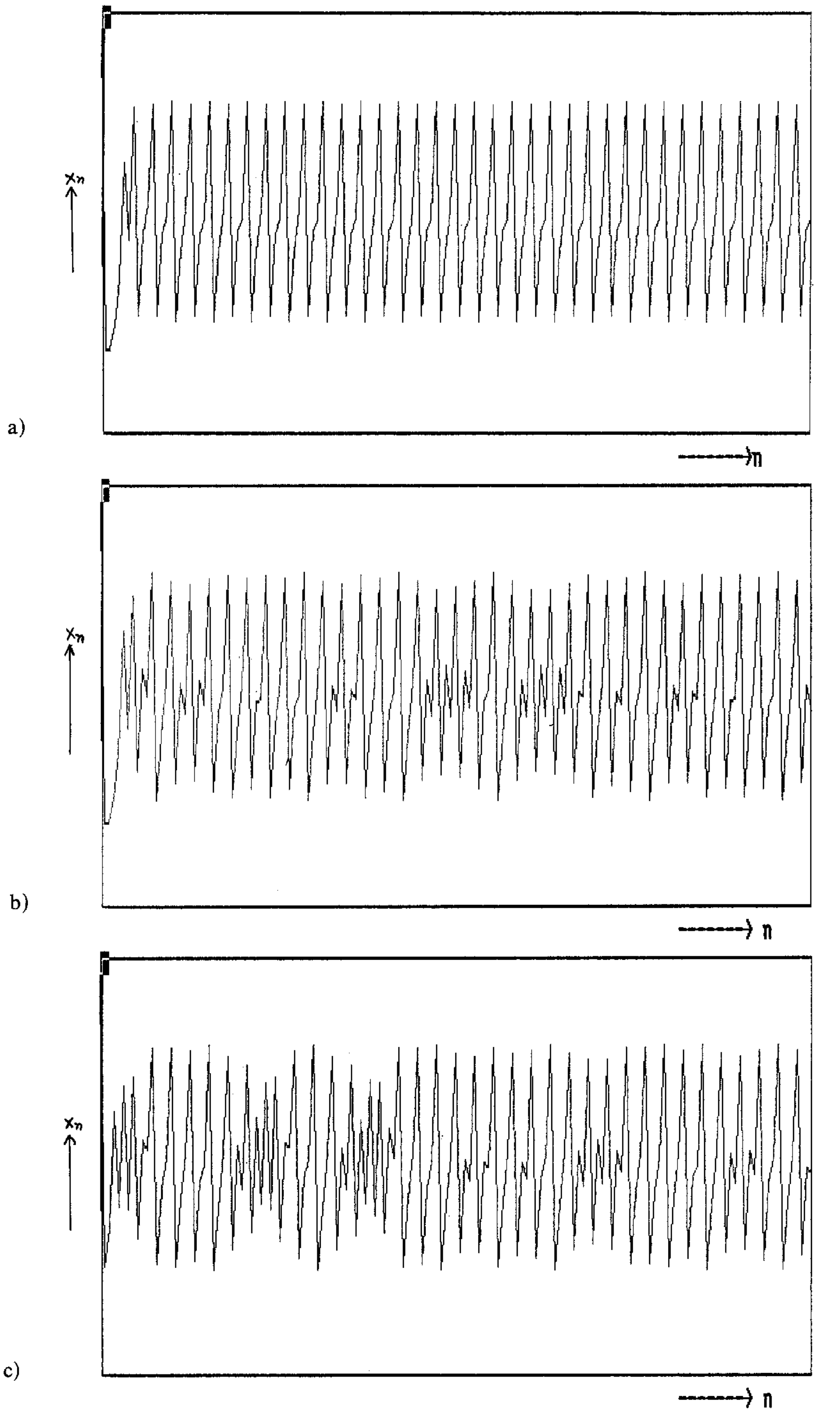


Fig. 2. Intermittency curve for the modulated logistic map for (a) $\mu = 0.89$, (b) $\mu = 0.898$, (c) $\mu = 0.899$.

increased to 0.898 the pattern changes totally and we observe a series of periodic and chaotic regimes. Next we set μ to 0.899, whence the situation becomes more chaotic. These observations justify the observation that the periodic behavior ends at $\mu = 0.89$ and the chaotic situation starts after that.

We next explore the structure of stable and unstable manifolds away from the fixed point. For this analysis we convert this coupled system to the form

$$\begin{aligned} x_{n+1} &= y_n \\ y_{n+1} &= F(\mu, x_n, y_n) \end{aligned} \tag{5}$$

with F given as

$$F = \mu \frac{y_n^2}{x_n} \frac{1-y_n}{1-x_n} \frac{4x_n(1-x_n)-y_n}{x_n(1-y_n)} \tag{6}$$

We first set up the functional equations for the invariant curves (W_s) stable manifold and (W_u) unstable manifold. We seek them in the form

$$\begin{aligned} y &= f(x) \\ x &= g(y) \end{aligned} \tag{7}$$

where we have omitted the suffix n. It is then easy to observe that $f(x)$ satisfies

$$x^2(1-x)^2 f[f(x)] = \mu f^2(x)[1-f(x)][4x(1-x)-f(x)] \tag{8}$$

and the equation for g is

$$y = g\left\{\frac{\mu y^2}{g(y)} \frac{1-y}{1-g(y)} \frac{4g(y)[1-g(y)]-y}{g(y)[1-g(y)]}\right\} \tag{9}$$

If we now set (Feigenbaum, 1980)

$$\begin{aligned} x &= x^* + \bar{x} \\ y &= y^* + \bar{y} \end{aligned} \tag{10}$$

so that

$$y = f(x) = f(x^* + \bar{x}) = y^* + \sum_{k=1} \alpha_k \bar{x}^k \quad \text{(say)} \tag{11}$$

then from (11) we get

$$\begin{aligned} &\alpha_1^2(a^4 - 2a^3 + a^2) + \alpha_1\mu(-5a^2 + 16a^3 - 12a^4) \\ &\quad - \mu(8a^4 - 12a^3 + 4a^2) + 2a^2 - 6a^3 + 4a^4 = 0 \end{aligned} \tag{12}$$

whose roots are

$$\alpha_1^+ = 2(2\mu - 1)$$
$$\alpha_1^- = \frac{2\mu - 1}{\mu} \tag{13}$$

For a similar analysis of $x = g(y)$ we set

$$x = g(y) = g(y^* + \bar{y}) = x^* + \sum_{k=1} \beta_k \bar{y}^k \tag{14}$$

The equation for β_1 is

$$2\beta_1^2(2\mu - 1)^2 - \beta_1(2\mu - 1)(2\mu + 1) + \mu = 0 \tag{15}$$

whence β_1 has two values,

$$\beta_1^+ = \frac{\mu}{2\mu - 1}; \qquad \beta_1^- = \frac{1}{2(2\mu - 1)} \tag{16}$$

Higher-order coefficients α_2, β_2, etc., can also be determined from the expansion of the functional equation. So finally we set

$$\bar{y} = \alpha_1 \bar{x} + \alpha_2 \bar{x}^2 + \alpha_3 \bar{x}^3 + \cdots$$
$$\bar{x} = \beta_1 \bar{y} + \beta_2 \bar{y}^2 + \beta_3 \bar{y}^3 + \cdots \tag{17}$$

It is interesting to note that in each case the lowest order terms yield the fixed point (x^*, y^*) as a check for consistency.

The equations for α_2, β_2 are

$$\alpha_2\{(\alpha_1 + \alpha_1^2)a^2(1-a)^2 + \mu[a^2(1-a) + a^2(3-4a)(3a-2)]\}$$
$$= \mu\{-4a^2(1-a) + (4 - 8a - \alpha_1)[2a\alpha_1(1-a) - \alpha_1 a^2]$$
$$+ (3a - 4a^2)(1-3a)\alpha_1^2\}$$

and

$$\beta_2\left\{\left[\frac{4\mu}{1-a}(2a\beta_1 - \beta_1 + 2 - 3a) - \frac{\mu}{(1-a)^2}(3 - 4a - 4\beta_1 a - 2\beta_1)\right]^2\right.$$
$$\left. + \frac{4\mu\beta_1}{1-a}(2a-1) - \frac{2\mu\beta_1}{(1-a)^2}(2a-1)\right\}$$
$$= -\frac{4\mu}{a^2(1-a)^2}[\beta_1^2(2a-1)(2a-3a^2) + \beta_1 a(1-a)(1-3a) - \beta_1^3 a^2(1-a)]$$
$$+ \frac{\mu}{a(1-a)^3}[2\beta_1^2(1-2a)a + 6\beta_1^2 \times (1-a)(2a-1)$$
$$+ 3\beta_1(1-a)^2 - 3\beta_1 a(1-a) - 4\beta_1^3 a(1-a)] \tag{18}$$

In this way one can determine the coefficients α_i, β_i as required, by keeping terms cubic in $(\bar{x}, \bar{y})$ in equations (11) and (14). In Figure 3a we have fixed

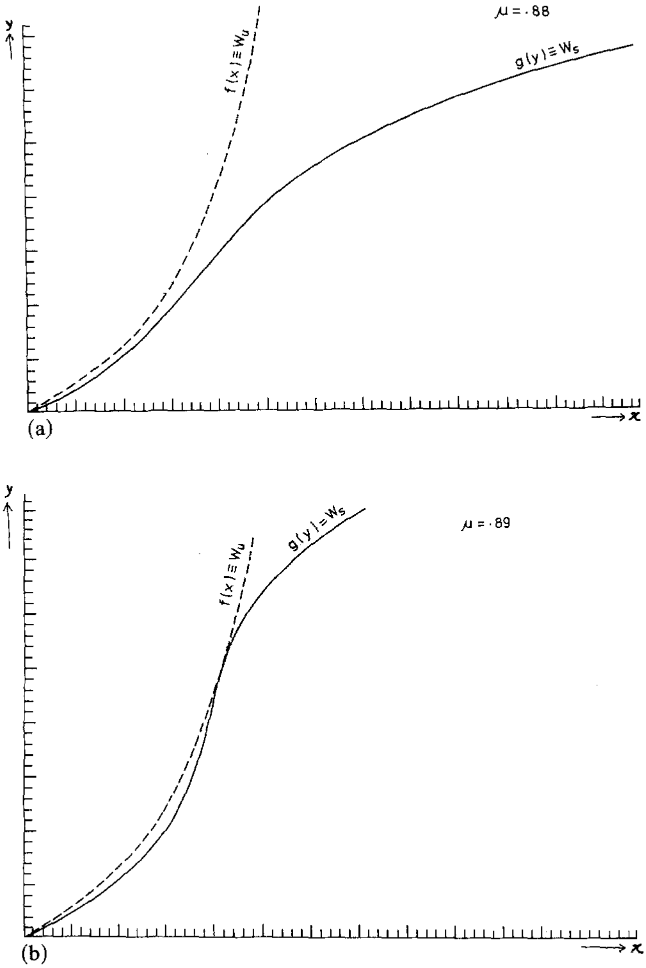
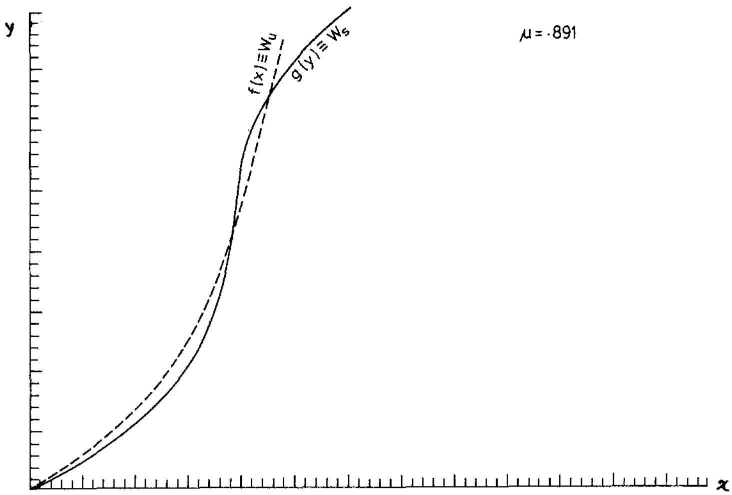


Fig. 3. Diagrams for the stable and unstable manifolds W_s and W_u for various values of the parameter μ. (a) $\mu = 0.88$, (b) $\mu = 0.89$ (note the homoclinic tangency between the stable and unstable manifolds for the critical value of μ), and (c) $\mu = 0.891$.

$\mu = 0.88$; the curves for $f(x) = W_u$ and $g(y) = W_s$ do not touch. In Figure 3b the value of μ is fixed at $\mu = 0.89$, and the curves $f(x)$ and $g(y)$ just touch. In Figure 3c, μ is further increased to 0.891, and the curves cross each other. These observations further confirm the observations regarding the zone of periodicity and chaos in the modulated logistic map.

Lastly we consider the problem of the determination of Lyapunov numbers. Meyer (1986) and Berry (1978) determine the form of the region of the attractor set. Attractors can exhibit a wide variety of shapes and it



(c)

Fig. 3. Continued.

is to be expected that the complexity of these shapes will be related in some way to the relative amount of stretching and compression. For a two-dimensional discrete map

$$\begin{aligned} x_{n+1} &= h(x_n, y_n) \\ y_{n+1} &= k(x_n, y_n) \end{aligned} \tag{19}$$

we can linearize about some exact orbit

$$(x_0, y_0) \to (x_1, y_1) \to \cdots (x_n, y_n) \to \cdots$$

to obtain the error propagation equation

$$\delta X_{n+1} = A_n \delta X_n \tag{20}$$

as in equation (4), where

$$\delta X_n = \begin{pmatrix} \delta x_n \\ \delta y_n \end{pmatrix}$$

with the Jacobi matrix

$$A_n = \begin{pmatrix} \partial h/\partial x_n & \partial h/\partial y_n \\ \partial k/\partial x_n & \partial k/\partial y_n \end{pmatrix} \tag{21}$$

is evaluated for each iteration at the point (x_n, y_n) on the exact trajectory. It is useful to think of the solution of the equation

$$\delta X_n = A_{n-1} A_{n-2} \ldots, \qquad A_0\, \delta X_0 = J_n\, \delta x_0$$

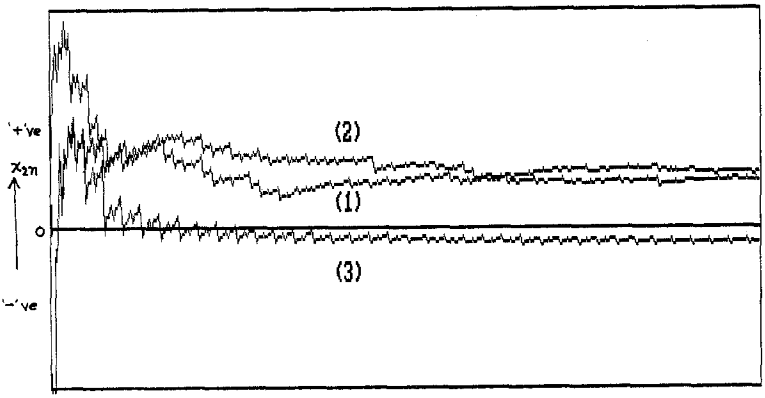


Fig. 4. Lyapunov coefficient for the modulated logistic map for (1) $\mu = 0.9$, $x_0 = \lambda_0 = 0.4$; (2) $\mu = 0.9$, $x_0 = \lambda_0 = 0.7$; (3) $\mu = 0.89$.

in terms of a matrix J_n which is composed of the product $A_0, A_1 \cdots A_{n-1}$. The eigenvalues of J_n vary with n, as do the eigenvectors, and are obtained by solving

$$\det[J_n - \sigma(n)I] = 0 \tag{22}$$

Lyapunov exponents χ_1, χ_2 are then defined by assuming that

$$\sigma_i(n) \to e^{n\chi_i} \qquad \text{as} \quad n \to \infty \tag{23}$$

so that $\chi_i \sim (1/n) \ln \sigma_i(n)$ for n large, so the stability of the system is controlled by the sign of χ_1, χ_2. If $\chi_1 + \chi_2 = 0$, then the mapping is an area-preserving one. On the other hand, if both of the coefficients are negative, then the system is stable. And if the larger of the pair (χ_1, χ_2) becomes positive, then it turns out to be unstable. Computational results for the present situation are displayed in Figure 4. It is easily seen that at $\mu = 0.89$ the Lyapunov index (in our case the larger of the two) remains negative. But as the value of μ is increased further to $\mu = 0.9$ (the initial value of the iteration being $x_0 = \lambda_0 = 0.4$) and in another situation when $\mu = 0.9$ but the starting value is $x_0 = \lambda_0 = 0.7$, the Lyapunov index converges but remains positive, showing the instability of the system beyond the parameter value $\mu = 0.89$.

3. CONCLUSION

We have explored the zones of periodicity and chaos in a logistic map whose original parameter follows a second nonlinear map with μ as the parameter. We have evidence that $\mu = 0.89$ is a critical value, because the bifurcation, intermittency, the Lyapunov index, and the stable-unstable manifold tangency all show a transition after $\mu = 0.89$.

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